Differential Geometry 2024, M.Math 2nd Year

Marks -100

8th November, 2024

1. Verify that the following statements are true or false (without justification) (2+2+2+2+2=10 points).
(i)There exists a 2-form α on $\mathbb{R}^4$ such that $\alpha \wedge \alpha \neq 0$.
(ii) There exists a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ and a regular value c of f such that Mobius Strip M can be expressed as $M = f^{-1}(c)$.
(iii)$\mathbb{R}^2$ and $\mathbb{S}^1 \times \mathbb{R}$ have same Gauss curvature.
(iv) There exists a compact minimal surface without boundary.
(v)The Euler characteristic of a tetrahedron in $\mathbb{R}^3$ is 2.

2. (10 points) Let V be a finite dimensional vector space and $f, g : V \to \mathbb{R}$ be two non-zero linear functionals such that $\ker(f) = \ker(g)$. Show that there exists a non-zero constant c such that $g = cf$.

3. (10 points) Compute $f^*\omega$ where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$f(x, y, z) = (xy - xz, xyz)$$

and $w = xdy \wedge dz + ydz \wedge dx + zdz \wedge dx$.

4. (10 points) Let $p \in \mathbb{S}^2$ and $v \in T_p\mathbb{S}^2 \subset \mathbb{R}^3$ be a unit vector. Then the curve $\gamma : [0, 2\pi] \to \mathbb{S}^2$ defined by

$$\gamma(t) = (p \cos t + v \sin t)$$

is a geodesic in $\mathbb{S}^2$.

5. (10 points) Let $S \subset \mathbb{R}^3$ be a regular surface without boundary and $p, q \in S$. Let α be a smooth parametrized curve from p to q. Then the parallel transport $P_\alpha : T_pS \to T_qS$ along α is an isometry; namely, P_α preserves the inner product.

6. (10+10+5=25 points) Let $S \subset \mathbb{R}^3$ be a compact non-empty regular surface without boundary and K be the Gauss curvature of S.
(a) Show that the Gauss map $S \to \mathbb{S}^2$ is onto.
(b) Show that if the Gauss map is injective, then $K \geq 0$.
(c) Is the Gauss map restricted to $S_+ := \{x \in S \mid K(x) \geq 0\}$ is onto? Justify your answer.

7. (10+5=15 points) Give an example of a 2-form ω on $\mathbb{R}^3$ such that if $\mathbb{S}^2$ is the unit sphere, the integral $\int_{\mathbb{S}^2} \omega \neq 0$. Prove that ω cannot be exact on $\mathbb{S}^2$.

8. (10 points) Let M be a closed oriented manifold of dimension $4k + 2$ and $\omega_1, \omega_2, \eta_1, \eta_2$ be closed $2k + 1$ forms on M such that

$$\omega_1 - \omega_2 = d\omega, \ \eta_1 - \eta_2 = d\eta,$$

for some $2k$ forms ω and η. Prove that

$$\int_M \omega_1 \wedge \eta_1 = \int_M \omega_2 \wedge \eta_2.$$